Section 10.4

The Divergence Test

Analysis of a given series begins with a simple question: does the series converge? If the answer is no, the series diverges and there is no further analysis required. If the answer is yes, the series converges and it may be possible to find an estimate for its value.

Theorem 10.4.1. Divergence Test
If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the series diverges.

Example 10.4.2.

1. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k+1}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{1+3^k}{2^k}$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$

Theorem 10.4.3. Integral Test
Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$ for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. In case of convergence, the value of the integral is NOT equal to the value of the series.

Example 10.4.4. Determine convergence of the following series.

1. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{\sqrt{2k-5}}$

2. $\sum_{k=1}^{\infty} \frac{1}{k^p}$

Theorem 10.4.5. Convergence of p-series
The *p-series* $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Estimating Series' Values

Definition 10.4.6.
The *remainder* is the error in approximating a series by the n^{th} partial sum.

$$R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = a_{n+1} + a_{n+2} + a_{n+2} + \cdots .$$

Theorem 10.4.7. Estimating R_n
Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$. Let $\sum_{k=1}^{\infty} a_k = S$. Then the remainder $R_n = S - S_n$ is bounded according to

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx.$$

This gives the following bounds on S:

$$L_n = S_n + \int_{n+1}^{\infty} f(x)\,dx < S < S_n + \int_{n}^{\infty} f(x)\,dx = U_n.$$

Example 10.4.8. What n guarantees an approximation of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ within 10^{-3} of the exact value?

Example 10.4.9. Evaluate $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\right)^2$

Example 10.4.10. Determine whether the following series converge or diverge.

1. $\displaystyle\sum_{k=1}^{\infty}\ln\left(\frac{2k^6}{1+k^6}\right)$

2. $\displaystyle\sum_{k=0}^{\infty}\frac{10}{k^2+9}$

3. $\displaystyle\sum_{k=2}^{\infty}\frac{1}{e^k}$

4. $\displaystyle\sum_{k=1}^{\infty}\frac{k}{\sqrt{k^2+1}}$

5. $\displaystyle\sum_{k=1}^{\infty}\frac{2^k+3^k}{4^k}$

6. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$